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Critical Exponents of Random Ising-Like Systems in General Dimensions

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Critical exponents of weakly dilute Ising-like systems are computed for noninteger space dimensionalities in the range $2 \leqslant d \leqslant 4$. The calculations are performed in the framework of the Callan–Symanzik field-theoretic approach. Two-loop renormalization group functions are obtained as renormalized perturbation theory series expansions directly in noninteger dimensions. The values of the critical exponents are estimated with the use of the two-variable Borel resummation method.

KEY WORDS: Critical phenomena; disordered spin systems; Ising model; field-theoretic approach in general dimensions.

INTRODUCTION

Much attention has been devoted recently to the investigation of different model lattice systems at noninteger space dimensionalities. The treatment of the critical behavior of spin systems at noninteger dimensions, besides being of purely academic interest, has several other attractions. There exist models where new phenomena can appear at noninteger space dimensionalities. As an example, the random-anisotropy model (RAM)[(1)] can be mentioned. It was shown[(2)] that in the case of infinite anisotropy there exists a competition between the space dimensionality and the number of spin components leading to either ferromagnetic or spin-glass ordering starting from some (noninteger) space dimensionality. The noninteger dimensionalities are widely used in the theory of fractals. The detailed analysis of fractal lattices has led to the controversial conjecture that some fractal lattices could interpolate the standard regular lattices in noninteger dimensions. Finally, in the investigation of different model systems the

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variations of dimensionalities of the order parameter and space can be used to link the results to the exact ones or to the results of other approximate calculational methods.[3-5]

Although for regular isotropic models the values of critical exponents for noninteger space dimensionalities $2 \leqslant d \leqslant 4$ are known with sufficiently high accuracy,[5] less is known in the case of anisotropic and random model systems. To our knowledge, the only paper where the critical exponents of the anisotropic cubic model and the dilute Ising model are calculated for $2.8 \leqslant d \leqslant 4$ is that of Newman and Riedel.[3] The aim of the present paper is to compute the critical exponents of random Ising-like systems for noninteger space dimensionalities in the range $2 \leqslant d \leqslant 4$, using the field-theoretic approach. We employ the Callan–Symanzik massive field theory framework within the two-loop approximation.

1. STATEMENT OF THE PROBLEM AND THE RENORMALIZATION-GROUP FUNCTIONS

The site-diluted random classical m-component spin system is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{i,j} \mathscr{J}_{ij} c_i c_j \mathbf{s}_i \mathbf{s}_j \tag{1.1}$$

where $\mathbf{s}_i$ and $\mathbf{s}_j$ are m-component classical spins located at the lattice sites i and j; $\mathscr{J}_{ij}$ is the translationally-invariant short-range ferromagnetic exchange interaction potential; and c_i and c_j are the occupation numbers, the random variables having for each site i the probability distribution

$$P(c_i) = c\delta(c_i - 1) + (1 - c)\,\delta(c_i)$$

c is the concentration of the lattice sites occupied by the spins. The summation in (1.1) is over all the sites of a regular d-dimensional lattice.

The critical behavior of quenched dilute[6] m-vector model (1.1) is governed by the translationally-invariant effective field Hamiltonian of the form

$$\mathscr{H} = \int d^d r \left\{ \frac{1}{2} \sum_{\alpha=1}^{n} [(\nabla \boldsymbol{\phi}^\alpha)^2 + m_0^2 |\boldsymbol{\phi}^\alpha|^2] + \frac{u_0}{4!} \left(\sum_{\alpha=1}^{n} |\boldsymbol{\phi}^\alpha|^2 \right)^2 + \frac{v_0}{4!} \sum_{\alpha=1}^{n} (|\boldsymbol{\phi}^\alpha|^2)^2 \right\} \tag{1.2}$$

in the replica limit $n \to 0$.[7] Here each vector field $\boldsymbol{\phi}^{\alpha} \equiv \boldsymbol{\phi}^{\alpha}(r)$ with $\alpha = 1,..., n$ has m components $(\phi^{\alpha,1},..., \phi^{\alpha,m})$. The quantity m_0^2 is the bare mass, which is a linear function of temperature. $u_0 < 0$ and $v_0 > 0$ are the bare coupling constants.

As is well known, the heuristic Harris criterion[8] gives the qualitative critical behavior of disordered spin systems. Namely, it predicts that the critical exponents of a dilute system will be the same as in the pure one if the specific heat exponent α_{pure} of the pure system is negative. If, on the other hand, $\alpha_{\text{pure}} > 0$, one can expect that disorder will lead to some new behavior. This picture is confirmed by the renormalization group (RG) calculations. From the very beginning the critical behavior of random spin systems was investigated by different RG methods in the vicinity of space dimensionality $d = 4$.[7,9–14] The critical exponents were obtained as rather short series in powers of $\varepsilon = 4 - d$ (in the case of the random Ising model, $m = 1$, the expansion parameter is[11] $\varepsilon^{1/2}$). The extrapolation of such series expansions toward large values of ε, say $\varepsilon = 1$ or $\varepsilon = 2$, could not lead to reliable numerical estimates. That is why papers appeared where the RG equations were treated directly at three or two dimensions following the approach proposed for pure systems by Parisi.[15] The appropriate renormalized perturbation series expansions combined with the asymptotic series resummations led to rather accurate values of the critical exponents for $d = 3$.[16–19] In the case of $d = 2$, with the use of the fermion representation for the dilute Ising model, it was shown that its critical behavior is identical to that of the pure Ising model, apart from some logarithmic factors.[19,20] Also, the critical behavior of anisotropic and random spin systems was treated by the scaling-field[3] and the approximate RG[21–23] methods.

In order to investigate the critical behavior of the effective Hamiltonian (1.2) at general space dimensions d, we use the standard procedure of renormalization of the one-particle irreducible vertex functions $\Gamma^{(L,N)}(p_1,..., p_L;\ k_1,..., k_N;\ m_0^2, u_0, v_0; d)$ at zero external momenta and nonzero mass.[24] Asymptotically, close to the critical point, the renormalized vertex functions $\Gamma_R^{(N)}(\{k_j\}; m^2, u, v; d)$ satisfy the homogeneous Callan–Symanzik equation[24]

$$\left[m\frac{\partial}{\partial m} + \beta_u(u, v)\frac{\partial}{\partial u} + \beta_v(u, v)\frac{\partial}{\partial v} - \frac{N}{2}\gamma_{\varphi}(u, v) \right] \Gamma_R^{(N)}(\{k_j\}; m^2, u, v; d) = 0 \qquad (1.3)$$

Here u, v, and m are the renormalized coupling constants and mass. This equation may be treated, in principle, for arbitrary noninteger fixed space dimensionality d. The β functions are defined by the equations

$$\beta_u(u, v)\frac{\partial \ln u_0}{\partial u} + \beta_v(u, v)\frac{\partial \ln u_0}{\partial v} = d - 4$$
$$\beta_u(u, v)\frac{\partial \ln v_0}{\partial u} + \beta_v(u, v)\frac{\partial \ln v_0}{\partial v} = d - 4 \tag{1.4}$$

The function $\gamma_\phi(u, v)$ is defined as

$$\gamma_\phi(u, v) = \beta_u(u, v)\frac{\partial \ln Z_\phi(u, v)}{\partial u} + \beta_v(u, v)\frac{\partial \ln Z_\phi(u, v)}{\partial v} \tag{1.5}$$

where the renormalization constant Z_ϕ is given by

$$Z_\phi^{-1} = \frac{\partial}{\partial k^2}\Gamma^{(2)}(k; m_0^2, u_0, v_0; d)|_{k^2=0} \tag{1.6}$$

It is implied that the bare parameters m_0^2, u_0, and v_0 are expressed here in terms of renormalized ones.

The solution of the Callan–Symanzik equations yields the critical exponents in d dimensions.[24] At the fixed point (u^*, v^*), which is defined by the simultaneous zero of both functions β_u and β_v, the function $\gamma_\phi(u^*, v^*)$ gives the value of the Fisher exponent η. The correlation length exponent ν can be calculated from the consideration of the two-point vertex function with ϕ^2 insertion, $\Gamma^{(1,2)}(\{0\}; m_0^2, u_0, v_0; d)$. The massive field theory normalization condition for this vertex function implies the following definition of the renormalization constant $\bar{Z}_{\phi^2}$,

$$\bar{Z}_{\phi^2}^{-1} = \Gamma^{(1,2)}(\{0\}; m_0^2, u_0, v_0; d) \tag{1.7}$$

Using this relation, one calculates the γ function

$$\bar{\gamma}_{\phi^2}(u, v) = \beta_u(u, v)\frac{\partial \ln \bar{Z}_{\phi^2}^{-1}}{\partial u} + \beta_v(u, v)\frac{\partial \ln \bar{Z}_{\phi^2}^{-1}}{\partial v} \tag{1.8}$$

which at the fixed point gives the value of the critical exponent combination $2 - \nu^{-1} - \eta$. Knowing two critical exponents η and ν, one can compute the other ones using the familiar scaling relations.[24]

The explicit expressions for the β and γ functions corresponding to the effective Hamiltonian (1.2) at d dimensions in the two-loop approximation are as follows:

$$\beta_u(u, v) = -(4-d)\,u\left\{1 - u - \frac{12}{mn+8}v + \frac{8}{(m+8)^2}u^2[(5m+22)f(d) + (m+2)j(d)] + \frac{96}{(m+8)(mn+8)}uv\left[(m+5)f(d) + \frac{m+2}{6}j(d)\right] + \frac{24}{(mn+8)^2}v^2\left[(mn+14)f(d) + \frac{mn+2}{3}j(d)\right]\right\} \tag{1.9}$$

$$\beta_v(u, v) = -(4-d)\, v \left\{ 1 - v - 2\frac{m+2}{m+8} u + \frac{8}{(mn+8)^2} v^2 [(5mn+22) f(d) + (mn+2) j(d)] + 24 \frac{m+2}{(m+8)^2} u^2 \left[f(d) + \frac{1}{3} j(d) \right] + 96 \frac{m+2}{(m+8)(mn+8)} uv \left[f(d) + \frac{1}{6} j(d) \right] \right\}$$

$$\gamma_\phi(u, v) = -4(4-d) \left[\frac{m+2}{(m+8)^2} u^2 + \frac{mn+2}{(mn+8)^2} v^2 + \frac{2(m+2)}{(m+8)(mn+8)} uv \right] j(d) \tag{1.10}$$

$$\bar{\gamma}_{\phi^2}(u, v) = (4-d) \left\{ \frac{m+2}{m+8} u + \frac{mn+2}{mn+8} v - \left[12 \frac{m+2}{(m+8)^2} u^2 + 12 \frac{mn+2}{(mn+8)^2} v^2 + 24 \frac{m+2}{(m+8)(mn+8)} uv \right] f(d) \right\} \tag{1.11}$$

Here the following notations are introduced:

$$f(d) = i_1(d)[D(d)]^{-2} - \frac{1}{2}$$
$$j(d) = i_2(d)[D(d)]^{-2} \tag{1.12}$$

where $D(d)$ is the one-loop Feynman integral

$$D(d) = (2\pi)^{-d} \int \frac{d^d k}{(k^2+1)^2} \tag{1.13}$$

and $i_1(d)$ and $i_2(d)$ are the following two-loop integrals:

$$i_1(d) = (2\pi)^{-2d} \int \frac{d^d q_1\, d^d q_2}{(q_1^2+1)^2 (q_2^2+1)[(q_1+q_2)^2+1]}$$
$$i_2(d) = (2\pi)^{-2d} \frac{\partial}{\partial k^2} \int \frac{d^d q_1\, d^d q_2}{(q_1^2+1)(q_2^2+1)[(q_1+q_2+k)^2+1]} \bigg|_{k^2=0} \tag{1.14}$$

The renormalization-group functions given above were analysed by Jug[16] at $n=0$ for space dimensionalities $d=3$ and $d=2$. At $d=3$ the integral combinations (1.12) can be calculated analytically: $f(2) = 1/6$, $j(3) = -2/27$. Substituting these values into the formulas for the β and γ functions, one obtains the two-loop expressions for the RG functions of the mn-component anisotropic model in three dimensions.[25]

2. CALCULATION OF THE INTEGRALS AND RESUMMATION OF THE RG FUNCTIONS

To set up the numerical calculations of the β and γ functions, one needs to know the values $f(d)$ and $j(d)$, (1.12), as functions of the continuous parameter d. Using the familiar rules for the computation of Feynman integrals,[24] one obtains for the one-loop integral $D(d)$ in (1.13)

$$D(d)=2^{-d}\pi^{-d/2}\Gamma\left(2-\frac{d}{2}\right) \tag{2.1}$$

The two-loop integrals $i_1(d)$ and $i_2(d)$, (1.14), can be reduced to double integrals containing the space dimensionality d as a parameter. The required result is (note that the following representation is not unique)

$$\begin{aligned} f(d)&=\Gamma(\varepsilon)\left[\Gamma\left(\frac{\varepsilon}{2}\right)\right]^{-2}\int_0^1 dx\,\frac{x}{[x(1-x)]^{1-\varepsilon/2}} \\ &\quad\times\int_0^1 dy\,\frac{y^{\varepsilon/2}}{[x(1-x)(1-y)+y]^{\varepsilon}}-\frac{1}{2} \\ j(d)&=-\Gamma(\varepsilon)\left[\Gamma\left(\frac{\varepsilon}{2}\right)\right]^{-2}\int_0^1\frac{dx}{[x(1-x)]^{-\varepsilon/2}}\int_0^1 dy\,\frac{y^{\varepsilon/2}(1-y)}{[x(1-x)(1-y)+y]^{\varepsilon}} \end{aligned} \tag{2.2}$$

where $\varepsilon=4-d$, and $\Gamma(x)$ is the Euler gamma function. The last integrals can be computed numerically for any space dimensionality of interest. Some numerical values of $f(d)$ and $j(d)$ accurate to the fifth decimal figure are given in Table I. Figure 1 represents the dependence of these functions

Table I. Typical Numerical Values of Two-Loop Integrals

d	$f(d)$	$j(d)$	d	$f(d)$	$j(d)$
2.0	0.28129	−0.11463	3.0	0.16667	−0.07407
2.1	0.27042	−0.11105	3.1	0.15385	−0.06907
2.2	0.25949	−0.10740	3.2	0.14059	−0.06378
2.3	0.24846	−0.10367	3.3	0.12670	−0.05814
2.4	0.23733	−0.09985	3.4	0.11232	−0.05208
2.5	0.22606	−0.09593	3.5	0.09703	−0.04553
2.6	0.21464	−0.09188	3.6	0.08084	−0.03838
2.7	0.20303	−0.08769	3.7	0.06338	−0.03047
2.8	0.19119	−0.08335	3.8	0.04440	−0.02162
2.9	0.17908	−0.07882	3.9	0.02347	−0.01158

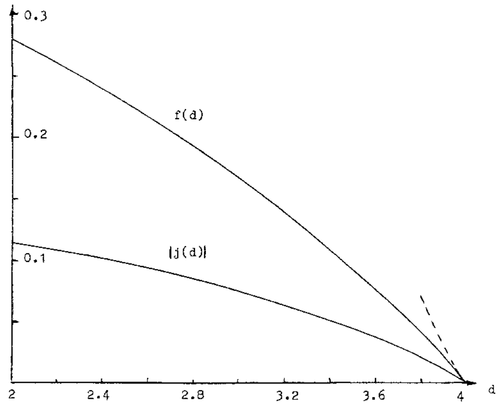

Fig. 1. Dependence of the two-loop integrals f and j on the space dimensionality d. The dashed curve gives the ε-expansion result for $f(d)$.

on the space dimensionality d. For comparison, the ε-expansion result accurate to the second order,

$$f(d)=\frac{\varepsilon}{4}+\frac{\varepsilon^2}{4}(3-J), \qquad J=0.7494$$

is shown on Fig. 1 by the dashed curve.

Thus, using the results of numerical calculations of two-loop integrals for noninteger dimensions, one can proceed in the framework of the Callan–Symanzik massive field theory approach directly in the dimensionalities of interest.

As is well known, renormalized perturbation series expansions like (1.9)–(1.11) are not convergent. In order to calculate the critical exponents proceeding from the RG functions (1.9)–(1.11), one has to apply to them a procedure of generalized summation of the series expansions. In the present paper we employ a two-variable resummation technique which is a simple generalization the of single-variable Padé–Borel method. Such a technique has been applied[16–19,25] for different models in integer space dimensions.

The starting point is a truncated power series expansion of the form

$$f(u,v)=\sum_{\substack{i,j\geqslant 0\\(i+j\leqslant n)}} c_{ij}u^i v^j \tag{2.3}$$

One constructs for this polynomial the Borel transform

$$F(ut,vt)=\sum_{\substack{i,j\geqslant 0\\(i+j\leqslant n)}} \frac{c_{ij}}{(i+j)!}(ut)^i(vt)^j \tag{2.4}$$

Then the Borel transform $F(x, y)$ is extrapolated by the rational approximant $F^A(x, y)$, and the resummed function $\tilde{f}(u, v)$ is represented by the integral

$$\tilde{f}(u, v) = \int_0^{\infty} F^A(ut, vt)\, e^{-t}\, dt \tag{2.5}$$

Within the present two-loop approximation the nondiagonal rational approximant[26] is used,

$$F^A(u, v) = \frac{1 + a_{10}u + a_{01}v + a_{11}uv}{1 + b_{10}u + b_{01}v} \tag{2.6}$$

which gives at $u = 0$ or $v = 0$ the usual [1/1] Padé approximant for the remaining variable. The coefficients a_{ij} and b_{ij} in (2.6) are

$$\begin{aligned} b_{10} &= -d_{20}/d_{10}, \qquad b_{01} = -d_{02}/d_{01} \\ a_{10} &= d_{10} + b_{10}, \qquad a_{01} = d_{01} + b_{01} \\ a_{11} &= d_{11} + b_{10}d_{01} + b_{01}d_{10} \end{aligned} \tag{2.7}$$

where $d_{kl} = c_{kl}/(k + l)!$, c_{kl} being the appropriate expansion coefficients of the function $f(u, v)$, (2.3). Substituting (2.6) into (2.5) and integrating over t, one obtains for the resummed function

$$\tilde{f}(u, v) = [xe^x E_1(x) - 1](1 - xy + a_{11}x^2uv) + a_{11}xuv + 1 \tag{2.8}$$

where $x = (b_{10}u + b_{01}v)^{-1}$, $y = a_{10}u + a_{01}v$, and[27]

$$E_1(x) = e^{-x} \int_0^{\infty} (x + t)^{-1}\, e^{-t}\, dt$$

It is obvious that several different ways of calculation of the resummed function $\tilde{f}(u, v)$ are possible. First, the representation of the Borel transform by the rational approximant is not unique. One can choose different forms of rational expressions with different numbers of terms in the numerator and denominator. In principle, it is necessary to try all the possibilities for such rationals and to build a table of results appropriate to different forms of rational approximants. The best convergence of results should be reached in the direction of the main diagonal of such a table. But in practice usually approximations are chosen which do not essentially complicate the calculations.

Moreover, the β functions having the structure $\beta_{u_i}(u, v) = -(4 - d)\, u_i f_i(u, v)$ can be resummed in different ways if one constructs the

Borel transforms for the whole functions or for the functions $f_i(u, v)$. The possibility of such arbitrariness within the resummation procedure was noted and discussed[28] in the case of the n-vector isotropic ϕ^4 effective Hamiltonian. For the three-dimensional mn-component anisotropic model (1.2) the resummations of $\beta_{u_i}(u, v)$ and of $f_i(u, v)$ lead to rather close numerical values of the critical exponents.[18] In the present paper the resummation of the whole β functions $\beta_{u_i}(u, v)$ is carried out. The explicit expression for the resummed β functions which will be used in the following is of the form [cf. (2.7), (2.8)]

$$\tilde{\beta}_{u_i}(u, v) = -(4-d)\, u_i x_i \{(1 - x_i y_i + a_{11}^{(i)} x_i^2 uv)[1 - xe^x E_1(x)] + y_i + a_{11}^{(i)}(2 - x_i)\, uv\} \tag{2.9}$$

The comparison between the two approaches to the resummation of the β functions is discussed elsewhere.[29] Whereas the difference between the fixed point coordinates calculated with the use of different resummations is significant, the critical exponents (especially for $d > 2.5$) do not differ strongly.

3. RESULTS

3.1. Pure Ising Model

In order to test the method of calculation in general dimensions, we start from the pure Ising model described by the effective Hamiltonian (1.2) with $v_0 = 0$ and $m = n = 1$. The two-loop RG functions for pure Ising system are obtained from (1.9)–(1.11) by putting $v = 0$, $m = 1$ (the coefficients at the powers of u do not depend on n). In this particular case the values of critical exponents for noninteger dimensions are known with high accuracy.[5] These values were computed by the resummation of five terms[30] of the ε-expansion series ($\varepsilon = 4 - d$).

The results of the numerical calculation of the pure Ising model critical exponents at different d are presented in Table II. Here the fixed point coordinates and the derivatives $\omega(d) = \tilde{\beta}'(u^*)$ exhibiting the stability of the fixed points are given as well. The critical exponent ν was calculated from the resummed function $1 - \bar{\gamma}_{\phi^2}(u) - \gamma_\phi(u)$, which corresponds at the fixed point to the value $\nu^{-1} - 1$.

The Fisher exponent η was computed by direct substitution of the fixed point coordinate u^* into the function $\gamma_\phi(u)$. The values of the other critical exponents were calculated using the familiar scaling laws.[24] Two-loop results for $d = 2$ and $d = 3$ which were given in ref. 16 are reproduced here. The comparison of our results with those of Le Guillou and Zinn-

Table II. Critical Indices of Pure Ising Model

d	ν	η	α	γ	β	δ	u^*	ω
2.0	0.981	0.200	0.038	1.766	0.098	19.000	2.424	0.922
2.1	0.924	0.168	0.060	1.693	0.124	14.671	2.318	0.909
2.2	0.874	0.141	0.077	1.625	0.149	11.903	2.219	0.893
2.3	0.830	0.118	0.091	1.652	0.173	10.004	2.125	0.874
2.4	0.791	0.098	0.099	1.506	0.197	8.638	2.038	0.851
2.5	0.758	0.081	0.105	1.455	0.220	7.606	1.955	0.827
2.6	0.727	0.062	0.110	1.405	0.242	6.796	1.876	0.798
2.7	0.700	0.055	0.110	1.362	0.264	6.152	1.802	0.767
2.8	0.675	0.044	0.110	1.320	0.285	5.635	1.731	0.732
2.9	0.652	0.035	0.106	1.283	0.305	5.203	1.662	0.695
3.0	0.632	0.028	0.104	1.246	0.324	4.836	1.597	0.654
3.1	0.614	0.021	0.097	1.214	0.344	4.526	1.533	0.609
3.2	0.596	0.016	0.093	1.182	0.362	4.263	1.472	0.561
3.3	0.580	0.012	0.083	1.155	0.381	4.030	1.412	0.509
3.4	0.566	0.008	0.076	1.127	0.398	3.829	1.353	0.453
3.5	0.553	0.006	0.066	1.102	0.416	3.648	1.295	0.392
3.6	0.540	0.003	0.056	1.078	0.433	3.491	1.237	0.327
3.7	0.529	0.002	0.043	1.057	0.450	3.347	1.180	0.256
3.8	0.518	0.001	0.031	1.035	0.466	3.219	1.121	0.179
3.9	0.509	0.000	0.015	1.018	0.484	3.105	1.062	0.094

Justin[5] is illustrated in Figs. 2 and 3. Here the exponents ν and η ($\ln \eta$) are plotted as functions of d and the best estimates[5] are marked by crosses. One can see that in the region $d \gtrsim 2.8$ our results practically coincide with those of ref. 5. As d is decreased, a difference appears. For example, at $d = 2$ the value of ν differs from the exact one, $\nu = 1$, by approximately 0.02.

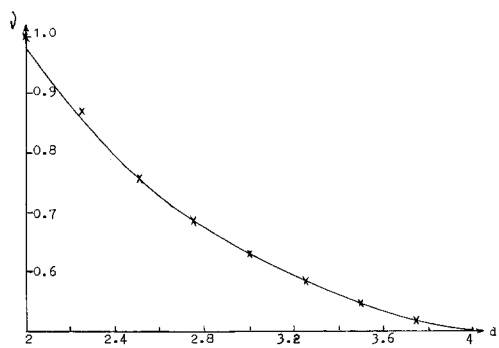

Fig. 2. The critical exponent ν of the pure Ising model as a function of d. The crosses give the results of ref. 5.

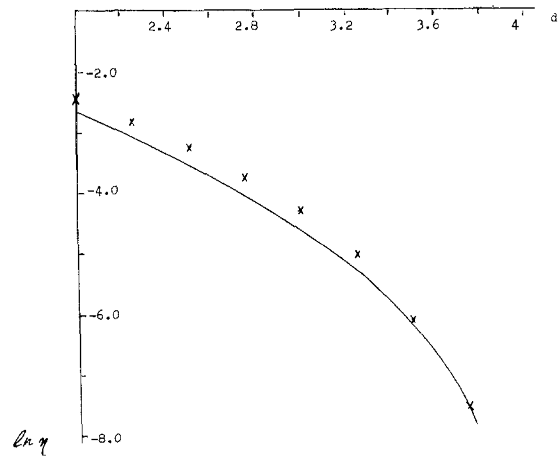

Fig. 3. The critical exponent η of the pure Ising model as a function of d. The crosses give the results of ref. 5.

3.2. Dilute Ising Model

The results of the previous subsection suggest that the application of the resummation procedure to the two-loop RG functions of Section 2 could be expected to produce reasonable critical exponent estimates for more complicated systems, one of them being the randomly dilute m-vector model.

As was already noted, the critical behavior of the dilute m-vector system is given by the effective Hamiltonian (1.2) at $n \to 0$. In this case the range of the renormalized coupling constants $u > 0$ and $v \leqslant 0$ is of interest. The fixed points defined by the simultaneous zero of both resummed β functions (1.9) are stable and determine the critical behavior of the system if the eigenvalues b_1 and b_2 of the stability matrix $B = \partial \tilde{\beta}_{u_i} / \partial u_j |_{u^*, v^*}$ are positive (or possess positive real parts if complex). It is well known that for large enough values of m, $m > m_c$, the isotropic ($u^* \neq 0, v^* = 0$) fixed point is stable and the critical behavior of a random system coincides with that of the corresponding pure system. When the number of the order parameter components decreases, starting from the marginal value m_c, the pure fixed point becomes unstable and the crossover to the mixed fixed point ($u^* \neq 0, v^* \neq 0; v^* < 0$) occurs. This last is appropriate for the description of new, "random" critical behavior. As was mentioned above, this qualitative picture agrees with the Harris criterion.[8]

Within the present two-loop approximation, $m_c = 2.01$ for $d = 3$, and $m_c = 1.19$ for $d = 2$.[16] Higher-order calculations lead to $m_c < 2$ for $d = 3$.[28] The best theoretical estimate of m_c in three dimensions is[31]

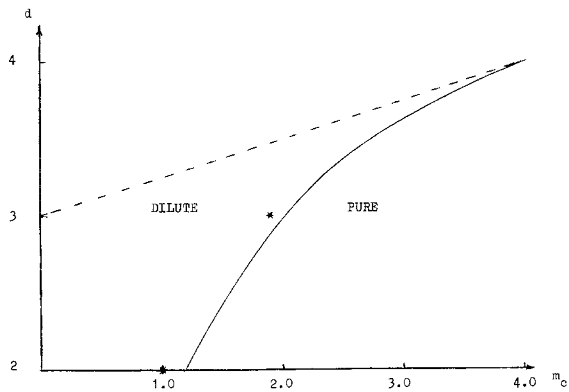

Fig. 4. The results for m_c as a function of d. The asterisks indicate the exact $(d=2)$[20] and most accurate $(d=3)$[31] values.

$m_c = 1.945 \pm 0.002$. In two dimensions the exact value is $m_c = 1$.[20] Here (see Fig. 4) we present the results of numerical calculations of m_c in the range $2 \leqslant d \leqslant 4$. For $d=4$, we have $m_c = 4$ (recall that the second order of the ε expansion yields $m_c = 4 - 4\varepsilon$).[7] For $d=3$ and $d=2$ the results of ref. 16 are reproduced. From Fig. 4 one can see that in the range of space dimensionalities $3 < d < 4$ a new critical behavior should appear, under dilution, for integer values of m, $m=2$ and $m=3$.

Table III. Critical Exponents of Dilute Ising Model

d	ν	η	α	u^*	v^*	b_1	b_2
2.0	1.012	0.198	−0.024	2.562	−0.130	0.933	0.096
2.1	0.960	0.167	−0.016	2.506	−0.170	0.916	0.126
2.2	0.914	0.141	−0.011	2.459	−0.210	0.892	0.155
2.3	0.873	0.119	−0.007	2.421	−0.252	0.863	0.185
2.4	0.837	0.100	−0.009	2.391	−0.294	0.827	0.216
2.5	0.804	0.833	−0.010	2.370	−0.338	0.784	0.248
2.6	0.774	0.069	−0.012	2.357	−0.384	0.733	0.284
2.7	0.747	0.057	−0.017	2.352	−0.433	0.670	0.326
2.8	0.722	0.047	−0.022	2.356	−0.486	0.584	0.386
2.9	0.699	0.038	−0.027	2.371	−0.542	0.469[a]	0.469[a]
3.0	0.678	0.031	−0.034	2.396	−0.605	0.450[a]	0.450[a]
3.1	0.658	0.024	−0.040	2.436	−0.676	0.429[a]	0.429[a]
3.2	0.640	0.019	−0.048	2.492	−0.756	0.405[a]	0.405[a]
3.3	0.622	0.014	−0.053	2.572	−0.852	0.378[a]	0.378[a]
3.4	0.606	0.010	−0.060	2.683	−0.968	0.344[a]	0.344[a]

[a] Complex. Real part is given.

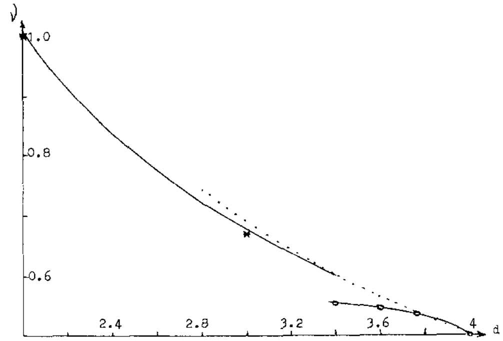

Fig. 5. The critical exponent ν of the dilute Ising model as a function of d. The dotted curve shows the results of ref. 3, and the open circles, the results of the resummation of the resummation of the $\varepsilon^{1/2}$ expansion. The asterisks give the exact $(d=2)$[20] and most accurate $(d=3)$[19] values.

Now we return to the most interesting case, $m=1$, the random Ising model, where the new critical behavior is observed in the whole range $2<d<4$. Table III gives the values of the critical exponents of d-dimensional random Ising systems, as well as the mixed fixed point coordinates u^*, v^* and the stability matrix eigenvalues b_1, b_2. The smooth dependences of $\nu(d)$ and $\eta(d)$ [$\ln \eta/d$)] are plotted in Figs. 5 and 6 by solid lines. The

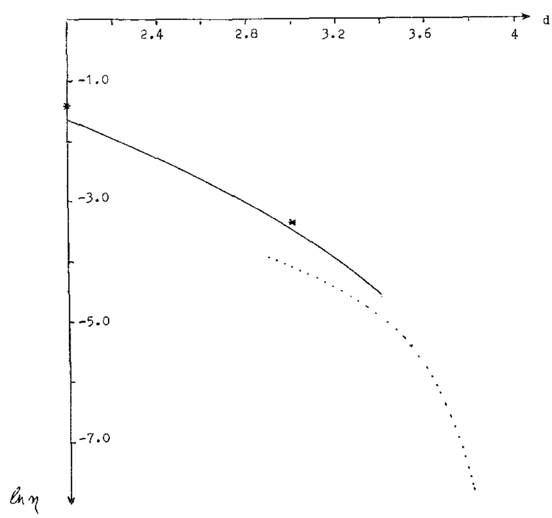

Fig. 6. The critical exponent η of the dilute Ising model as a function of d. The dotted curve shows the results of ref. 3. The asterisks give the exact $(d=2)$[20] and most accurate $(d=3)$[19] values.

approximations used in the present calculations lead to a limitation on the large values of d in the case of the dilute Ising model. The reason is that at large dimensionalities $d \gtrsim 3.5$ a pole appears in the rational approximants $\beta^A(u^*t, v^*t)$ entering the integral defining the resummed β functions [see (2.5)–(2.6)], and the corresponding integrals become divergent. We excluded the approximants with positive real poles from consideration, and that is why only the results up to dimensionality $d = 3.4$ are presented. Again, the known two-loop results[16] for three and two dimensions are reproduced.

The results of the present paper are compared with other available data in Figs. 5 and 6. The results of scaling-field method calculations in the region $2.8 \leqslant d \leqslant 4$[3] are plotted by dotted curves. The limitation on the space dimensionalities on the left-hand side is caused by the fact that the truncated set of scaling-field equations considered in ref. 3 breaks down for the dimensionalities of space lower than 2.8. Also, the most accurate three-dimensional estimates[19] are shown in Figs. 5 and 6 by asterisks. Note that no reliable values of the critical exponents for dilute Ising systems could be obtained from the expansion in powers of $\varepsilon = 4 - d$ as was done for pure systems by Le Guillou and Zinn-Justin.[5] The reason is that the $\varepsilon^{1/2}$ expansions for the dilute Ising model are known at present only up to $O(\varepsilon)$ [up to $O(\varepsilon^{3/2})$ for η],[12,13] and also that the large-order behavior of the $\varepsilon^{1/2}$ expansion is not known. Nevertheless, we have tried to compute the critical exponents for the dilute Ising model in general dimensions from their $\varepsilon^{1/2}$ expansions using the resummation procedure described in Section 2, similar to what was done for the case $d = 3$.[32] The results for the exponent ν are plotted in Fig. 5 by open circles. They are reasonable only rather close to $d = 4$. One can see from Fig. 5 that the results of the present paper obtained by the field-theoretic approach for fixed noninteger dimensionalities are to be preferred at lower space dimensionalities.

For the three-dimensional dilute Ising model within the two-loop approximation the approach to the fixed point is oscillatory, and the eigenvalues of the stability matrix, b_1 and b_2, are complex. Higher-order calculations lead to positive real eigenvalues b_1 and b_2.[18,19] A similar situation takes place if one lowers the space dimensionality d remaining in the framework of the two-loop approximation (see Table III): b_1 and b_2, which were complex at $d = 3$, become real and positive near $d = 2.8$. One can see the situation when the properties of the fixed point can be controlled by a shift of the space dimension d, similar to the case involving the higher-order calculations. Also, if one appropriately lowers the dimension of space d, within the two-loop approximation one can see the qualitatively correct three-dimensional picture of the crossover from the pure to the mixed fixed point at $M = M_c < 2$.[31] This suggests that the exploration of the RG equation in general, noninteger space dimensionalities can provide information on real, three-dimensional complex models of disordered or anisotropic materials.

It is hard to expect that the results found here in the framework of the two-loop approximation would be extremely accurate. But some conclusions about their accuracy can be drawn. In the case of the theory with one coupling (considered in Section 3.1; pure Ising model) at $d=2$ our result $\nu(d=2)=0.981$ differs from the exact $\nu(d=2)=1.0$ by on the order 2%. With increase of d the accuracy increases as well and for the correlation length critical exponent, starting from $d=2.5$ it differs by less than 0.1% [compare data of Table II with those of Le Guillou and Zinn-Justin:[5] $\nu(d=2.5)=0.758\pm 5$, $\nu(d=3)=0.631\pm 15$). Typical HTS results are $\nu(d=3)=0.6305\pm 15$; 0.6300 ± 30; 0.6320 ± 10; 0.6295 ± 15 (see ref. 5 and references therein). The relative accuracy of the Fisher exponent η is worse (compare our η from Table II with that of ref. 5: $\eta(d=2.0)=0.25$, $\eta(d=2.5)=0.11\pm 1$, $\eta(d=3.0)=0.0375\pm 25$, as well as with HTS results, $\eta(d=3)=0.0360\pm 20$; 0.0390 ± 40; 0.0350 ± 10 (see ref. 5 and references therein)). This is caused by the fact that the absolute value of η is an order of magnitude less than that of ν, as well as that η was computed here by direct substitution of the fixed point coordinate into the appropriate RG function.

A similar picture can be observed for the model with two couplings. In the case considered in Section 3.2 of the dilute Ising model at $d=2$, our result $\nu(d=2)=1.012$ (Table III) differs from Onsager's by on the order 2%. At $d=3$ the two-loop result $\nu(d=3)=0.678$ (Table III) differs from the four-loop result of Mayer *et al.*,[19] $\nu(d=3)=0.6701$ by on the order 0.1%. As in the case of the pure Ising model, the relative accuracy of determination of η is worse [compare two-loop results $\eta(d=2)=0.198$, $\eta(d=3)=0.031$ with Onsager's $\eta(d=2)=1/4$ and the four-loop result $\eta(d=3)=0.0343$.[19]]. Let us give for comparison the most accurate value of m_c at $d=3$ (marked by an asterisk on Fig. 4), $m_c(d=3)=1.945\pm 0.002$[31] [compare with the two-loop result $m_c(d=3)=2.01$] and the exact value $m_c(d=2)=1$ [compare with the two-loop result $m_c(d=2)=1.19$].

It can be seen from the above data that the accuracy of the results obtained depends on d and in the case of the correlation length critical exponent for the dilute Ising model changes from 2% at $d=2$ to 0.1% at $d=3$. Such a dependence on the space dimensionality is caused by the different values of numerical coefficients entering the series to be resummed, namely, by the increase of the stable fixed point value as d decreases. Such a situation needs a separate analysis. The values obtained here of the critical exponents for the dilute Ising model at noninteger d are expected to be the most reliable in comparison with other data for noninteger d. The field-theoretic approach used directly in noninteger dimensions can obtain information about the critical behavior of systems of interest.

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REFERENCES

1. R. Harris, M. Plischke, and M. J. Zuckermann, New model for amorphous magnetism, *Phys. Rev. Lett.* **31**:160 (1973).
2. A. B. Harris, R. G. Caflisch, and J. R. Banavar, Random-anisotropy-axis magnet with infinite anisotropy, *Phys. Rev. B* **35**:4929 (1987).
3. K. E. Newman and E. K. Riedel, Cubic N-vector model and randomly dilute Ising model in general dimensions, *Phys. Rev. B* **25**:264 (1982).
4. J. C. Le Guillou and J. Zinn-Justin, Accurate critical exponents from the ε-expansions, *J. Phys. Lett.* **46**:L-137 (1985).
5. J. C. Le Guillou and J. Zinn-Justin, Accurate critical exponents for Ising like systems in non-integer dimensions, *J. Phys.* (Paris) **48**:19 (1987).
6. R. Brout, Statistical mechanical theory of a random ferromagnetic system, *Phys. Rev.* **115**:824 (1959).
7. G. Grinstein and A. Luther, Application of the renormalization group to phase transitions in disordered systems, *Phys. Rev. B* **13**:1329 (1976).
8. A. B. Harris, Effect of random defects on the critical behavior of Ising models, *J. Phys. C* **7**:1671 (1974).
9. A. B. Harris and T. C. Lubensky, Renormalization-group approach to the critical behaviour of random-spin models, *Phys. Rev. Lett.* **33**:1540 (1974).
10. T. C. Lubensky, Critical properties of random-spin models from the ε-expansion, *Phys. Rev. B* **11**:3573 (1975).
11. D. E. Khmelnitskii, Phase transition of the second kind in inhomogeneous bodies, *Zh. Eksp. Teor. Fiz.* **68**:1960 (1975).
12. C. Jayaprakash and H. J. Katz, Higher-order corrections to the $\varepsilon^{1/2}$-expansion of the critical behavior of the random Ising system, *Phys. Rev. B* **66**:3987 (1977).
13. B. N. Shalaev, Phase transition in a weakly disordered uniaxial ferromagnet, *Zh. Eksp. Teor. Fiz.* **73**:3201 (1977).
14. K. De'Bell and D. J. W. Geldart, Coefficients to $O(\varepsilon^3)$ of the mixed fixed point of the mn-component field model, *Phys. Rev. B* **32**:4763 (1985).
15. G. Parisi, Field-theoretic approach to second-order phase transitions in two- and three-dimensional systems, *J. Stat. Phys.* **23**:49 (1980).
16. G. Jug, Critical behaviour of disordered spin systems in two and three dimensions, *Phys. Rev. B* **27**:609 (1983).
17. I. O. Mayer and A. I. Sokolov, Critical indices of impure Ising model, *Fiz. Tverd. Tela* **26**:3454 (1984); On the critical behaviour of cubic crystals undergoing structural phase transitions, *Izv. Akad. Nauk SSSR* **51**:2103 (1987).
18. N. A. Shpot, Critical behaviour of the mn-component field model in three dimensions. II. Three-loop results; *Phys. Lett. A* **142**:474 (1989); On the critical behaviour of the

mn-component field model in three dimensions: Three-loop approximation, Preprint ITP-88-140R, Kiev (1988).

19. I. O. Mayer, A. I. Sokolov, and B. N. Shalaev, Critical exponents for cubic and impure uniaxial crystals: Most accurate theoretical values, *Ferroelectricity* **95**:93 (1989); I. O. Mayer, Critical exponents of the dilute Ising model from four-loop expansions, *J. Phys. A* **22**:2815 (1989).
20. B. N. Shalaev, Phase transition in two-dimensional ferromagnet with cubic anisotropy, *Fiz. Tverd. Tela* **30**:895 (1988); Correlation function and susceptibility of two-dimensional ferromagnet with cubic anisotropy, *Fiz. Tverd. Tela* **31**:93 (1989).
21. Yu. V. Holovatch, Phase transition in the anisotropic cubic model, *Ferroelect. Lett.* **8**:11 (1987); Critical behaviour of a model with non-spherical symmetry, Preprint ITP-86-143E, Kiev (1986).
22. Yu. V. Holovatch, Critical temperature of the site dilute model by the approximate renormalization group approach, *Ferroelect. Lett.* **11**:111 (1990); Pecularities of the critical behaviour of the dilute *m*-vector model in percolation region, Preprint 88-153E, Kiev (1988).
23. Yu. V. Holovatch and N. A. Shpot, Critical indices of random *m*-vector model by approximate renormalization group approach, *Acta Phys. Pol. A* **78**:369 (1990); Approximate renormalization group transformation for the dilute *m*-vector model, Preprint ITP-87-39E, Kiev (1987).
24. E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Field theoretical approach to critical phenomena, in *Phase Transitions and Critical Phenomena*, Vol. VI, C. Domb and M. S. Green, eds. (Academic Press, New York, 1976); D. J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).
25. N. A. Shpot, Critical behaviour of the *mn*-component field model in three dimensions, *Phys. Lett. A* **133**:125 (1988).
26. J. S. R. Chisholm, Rational approximants defined from double power series, *Math. Comput.* **27**:841 (1973).
27. M. Abramowitz and A. I. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (National Bureau of Standards, 1964).
28. E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Critical exponents for the *n*-vector model in three dimensions from field theory, *Phys. Rev. Lett.* **39**:95 (1977).
29. Yu. Holovatch and N. Shpot, Critical behaviour of dilute Ising-like systems in non-integer dimensions, Preprint ITP-90-48E, Kiev (1990).
30. S. G. Gorishny, S. A. Larin and F. V. Tkachov, ε-Expansion for critical exponents: The $O(\varepsilon^5)$ approximation, *Phys. Lett. A* **101**:120 (1984).
31. C. Bervillier, Estimate of a universal critical amplitude ratio from its ε-expansion up to ε^2, *Phys. Rev. B* **34**:8141 (1986).
32. N. A. Shpot, Equation of state and universal combinations of thermodynamic critical amplitudes of dilute Ising model, *Zh. Eksp. Teor. Fiz.* **98**:1762 (1990).